Notes on Extreme Value Theory

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Abstract

These notes introduce extreme value theory, focusing on maxima, domains of attraction, generalized extreme value limits, threshold exceedances, generalized Pareto approximations, tail index estimation, and extreme quantiles.

Contents

1 Motivation

Classical probability often studies averages, such as

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Extreme value theory studies maxima, minima, and tail events. For independent and identically distributed observations $X_1, \dots, X_n$, define the sample maximum

$$M_n = \max\{X_1, \dots, X_n\}.$$

The central question is: after suitable normalization, does M_n converge to a non-degenerate limiting distribution?

2 Distribution of the Maximum

Let $X_1, \dots, X_n$ be i.i.d. with distribution function F. Then

$$\mathbb{P}(M_n \le x) = \mathbb{P}(X_1 \le x, \dots, X_n \le x) = F(x)^n.$$

Without normalization, this distribution often degenerates. Therefore, one studies normalized maxima:

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \le x\right) = F(a_n x + b_n)^n,$$

where $a_n > 0$ and $b_n \in \mathbb{R}$ are normalizing constants.

3 Generalized Extreme Value Distribution

Definition 3.1 (GEV distribution)**.** *The generalized extreme value distribution is*

$$G_\xi(x) = \exp\left\{-\,(1 + \xi x)^{-1/\xi}\right\},$$

defined on the set where

$$1 + \xi x > 0.$$

For $\xi = 0$, the limiting form is

$$G_0(x) = \exp\{-e^{-x}\}.$$

The parameter ξ is called the shape parameter or extreme value index. It controls the heaviness of the tail.

4 Fisher–Tippett–Gnedenko Theorem

Theorem 4.1 (Extreme value limit theorem)**.** *Suppose there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that*

$$\frac{M_n - b_n}{a_n} \xrightarrow{d} Z$$

for a non-degenerate random variable Z. Then the distribution of Z belongs to the generalized extreme value family.

This theorem is the extreme-value analogue of the central limit theorem. The central limit theorem describes limits of normalized sums, while the Fisher–Tippett–Gnedenko theorem describes limits of normalized maxima.

5 Types of Extreme Value Limits

The GEV family unifies three classical cases.

5.1 Fréchet Type

Heavy-tailed distributions have $\xi > 0$. Examples include Pareto-type distributions. The upper endpoint is infinite.

5.2 Gumbel Type

Light-tailed distributions have $\xi = 0$. Examples include normal, exponential, and gamma-type distributions.

5.3 Weibull Type

Bounded-tail distributions have $\xi < 0$. The distribution has a finite upper endpoint.

6 Threshold Exceedances

Instead of block maxima, one may study exceedances above a high threshold u. Given $X > u$, define the excess

$$Y = X - u.$$

The conditional excess distribution is

$$F_u(y) = \mathbb{P}(X - u \le y \mid X > u) = \frac{F(u+y) - F(u)}{1 - F(u)},$$

for $y \ge 0$.

7 Generalized Pareto Distribution

Definition 7.1 (Generalized Pareto distribution)**.** *The generalized Pareto distribution is*

$$H_{\xi,\sigma}(y) = 1 - \left(1 + \frac{\xi y}{\sigma}\right)^{-1/\xi},$$

where $\sigma > 0$ *and*

$$1 + \frac{\xi y}{\sigma} > 0.$$

For $\xi = 0$*, the limiting form is*

$$H_{0,\sigma}(y) = 1 - \exp\left(-\frac{y}{\sigma}\right).$$

8 Pickands–Balkema–de Haan Theorem

Theorem 8.1 (Threshold exceedance theorem)**.** *For a large class of distributions, as the threshold* u *approaches the upper endpoint of the distribution, the conditional excess distribution* F_u *can be approximated by a generalized Pareto distribution:*

$$F_u(y) \approx H_{\xi,\sigma(u)}(y).$$

This result justifies the peaks-over-threshold method.

9 Extreme Quantiles

Let q_p denote a high quantile satisfying

$$\mathbb{P}(X > q_p) = p,$$

where p is small. Extreme quantile estimation aims to estimate q_p for probabilities p beyond the observed empirical range.

Using the GPD approximation above a threshold u, one can extrapolate into the tail:

$$\mathbb{P}(X > x) \approx \mathbb{P}(X > u) \left(1 + \frac{\xi(x-u)}{\sigma}\right)^{-1/\xi}.$$

Solving for x gives an estimator for high quantiles.

10 Tail Index

The parameter ξ determines tail behavior.

- If $\xi > 0$, the distribution is heavy-tailed.
- If $\xi = 0$, the distribution is light-tailed.
- If $\xi < 0$, the distribution has a finite upper endpoint.

Estimating ξ is therefore central in extreme value analysis.

11 References

References

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[3] Laurens de Haan and Ana Ferreira. *Extreme Value Theory: An Introduction*. Springer, 2006.