Notes on Probability Theory

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Abstract

These notes cover basic probability spaces, random variables, distributions, expectation, conditional expectation, convergence concepts, laws of large numbers, and central limit theorems.

Contents

1 Probability Spaces

Definition 1.1 (Probability space)**.** *A probability space is a triple*

$$(\Omega, \mathcal{F}, \mathbb{P}),$$

where Ω is the sample space, $\mathcal{F}$ is a sigma-algebra of events, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure satisfying:

(i) $\mathbb{P}(\Omega) = 1$*;*

(ii) $\mathbb{P}(A) \geq 0$ *for all* $A \in \mathcal{F}$*;*

(iii) if $A_1, A_2, \ldots$ *are disjoint events, then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

2 Random Variables

Definition 2.1 (Random variable)**.** *A random variable is a measurable function*

$$X : \Omega \to \mathbb{R}.$$

Measurability means that for every Borel set $B \in \mathcal{B}(\mathbb{R})$,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

3 Distribution Functions

Definition 3.1 (Cumulative distribution function)**.** *The distribution function of a random variable X is*

$$F_X(x) = \mathbb{P}(X \leq x), \quad x \in \mathbb{R}.$$

A distribution function is nondecreasing, right-continuous, and satisfies

$$\lim_{x \to -\infty} F_X(x) = 0, \qquad \lim_{x \to \infty} F_X(x) = 1.$$

4 Expectation

For a discrete random variable,

$$\mathbb{E}[X] = \sum_x x\mathbb{P}(X = x),$$

whenever the sum is well-defined.

For a continuous random variable with density f_X,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx,$$

whenever the integral is well-defined.

Definition 4.1 (Variance)**.** *The variance of X is*

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2].$$

Equivalently,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2.$$

5 Conditional Probability and Independence

Definition 5.1 (Conditional probability)**.** *If $\mathbb{P}(B) > 0$, then*

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Definition 5.2 (Independence)**.** *Events A and B are independent if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Random variables X and Y are independent if their generated sigma-algebras are independent.

6 Conditional Expectation

Conditional expectation may be understood as the best mean-square prediction of a random variable given available information.

Definition 6.1 (Conditional expectation). *Let X be integrable and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-sigma-algebra. A random variable Y is called the conditional expectation of X given $\mathcal{G}$ if:*

(i) Y is $\mathcal{G}$-measurable;

(ii) for every $G \in \mathcal{G}$,

$$\int_G Y \, d\mathbb{P} = \int_G X \, d\mathbb{P}.$$

We write

$$Y = \mathbb{E}[X \mid \mathcal{G}].$$

7 Modes of Convergence

Let X_n and X be random variables.

Definition 7.1 (Almost sure convergence). *We say X_n converges almost surely to X if*

$$\mathbb{P}\left(\{\omega : X_n(\omega) \to X(\omega)\}\right) = 1.$$

We write

$$X_n \xrightarrow{a.s.} X.$$

Definition 7.2 (Convergence in probability). *We say X_n converges in probability to X if for every $\varepsilon > 0$,*

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0.$$

We write

$$X_n \xrightarrow{p} X.$$

Definition 7.3 (Convergence in distribution). *We say X_n converges in distribution to X if*

$$F_{X_n}(x) \to F_X(x)$$

at every continuity point x of F_X. We write

$$X_n \xrightarrow{d} X.$$

8 Law of Large Numbers

Theorem 8.1 (Weak law of large numbers). *Let $X_1, X_2, \dots$ be independent and identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Then*

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \mu.$$

9 Central Limit Theorem

Theorem 9.1 (Central limit theorem). *Let $X_1, X_2, \dots$ be independent and identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Then*

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0, 1).$$

10 References

References

[1] Patrick Billingsley. *Probability and Measure*. Wiley.

[2] Rick Durrett. *Probability: Theory and Examples*. Cambridge University Press.